ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 258 - Introduction to Differential Equations

SECOND MIDTERM EXAMINATION

24.04.2017

Answer key

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 110 minutes

Question	Grade	Out of
1		12
2		20
3		20
4		20
5		13
6		20
Total		105

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Given that $y_1 = e^x$ is a solution of the equation $xy'' - (x+1)y' + y = 0$. Find a second linearly independent solution of the given equation.

$y_2 = e^x v \qquad y_2' = e^x v + e^x v' \; ; \; y_2'' = e^x v + 2e^x v' + e^x v''$

Putting these in the eqn, we get

$\cancel{xe^x v} + \cancel{2xe^x v'} + xe^x v'' - \cancel{xe^x v} - \cancel{xe^x v'} - \cancel{e^x v} - e^x v' + \cancel{e^x v} = 0$

$xe^x v'' + (x-1)e^x v' = 0$

$v'' + \frac{x-1}{x} v' = 0 \qquad$ Let $w = v'$

$w' = -\frac{x-1}{x} w$

$\frac{dw}{w} = (-1 + \frac{1}{x})\, dx$

$\ln w = -x + \ln x \implies w = xe^{-x}$

$v' = xe^{-x} \; ; \; v = \int xe^{-x}\, dx = -(x+1)e^{-x}$

So $y_2 = e^x[-(x+1)e^{-x}] = -(x+1) \quad$ or $y_2 = x+1$

Question 2. Find the general solution of $y''' - y'' = 6 + e^x$.

$r^3 - r^2 = 0 \qquad r^2(r-1) = 0 \qquad r=0,\ r=1$

$y_h = c_1 + c_2 x + c_3 e^x$

$y_p = Ax^2 + Bxe^x$

$y_p' = 2Ax + Be^x + Bxe^x$

$y_p'' = 2A + 2Be^x + Bxe^x$

$y_p''' = 3Be^x + Bxe^x$

Plug in the nonhomogeneous eqn

$3Be^x + \cancel{Bxe^x} - 2A - 2Be^x - \cancel{Bxe^x} = 6 + e^x$

$-2A = 6 \implies A = -3$

$B = 1$

$y = y_h + y_p = c_1 + c_2 x + c_3 e^x - 3x^2 + xe^x$

Question 3. Find the general solution of $y'' - 2y' + y = e^x \ln x$.

$r^2 - 2r + 1 = 0 \implies (r-1)^2 = 0$

$y_h = c_1 e^x + c_2 x e^x$

$y_p = v_1 e^x + v_2 x e^x$

$e^x v_1' + v_2' x e^x = 0$

$v_1' e^x + v_2' (x+1) e^x = e^x \ln x$

$$v_1' = \frac{\begin{vmatrix} 0 & xe^x \\ e^x \ln x & (x+1)e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix}} = \frac{-x\cancel{e^{2x}}\ln x}{\cancel{e^{2x}}} = -x\ln x$$

$$v_1 = \int -x\ln x\,dx = -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$$v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x \ln x \end{vmatrix}}{e^{2x}} = \frac{e^{2x}\ln x}{e^{2x}} = \ln x$$

$$v_2 = \int \ln x\,dx = x\ln x - x$$

$$y_p = -e^x \frac{x^2}{2}\ln x + \frac{x^2}{4}e^x + x^2 e^x \ln x - x^2 e^x$$

$$= \frac{1}{2}x^2 e^x \ln x - \frac{3}{4}x^2 e^x$$

$$y = y_h + y_p = c_1 e^x + c_2 x e^x + \frac{1}{2}x^2 e^x \ln x - \frac{3}{4}x^2 e^x$$