Equations of the form $y' = G(ax+by)$

1) Solve $\frac{dy}{dx} = (x+y)^{\frac{1}{2}} - 1$

Let $\vartheta = x+y$, then $\frac{d\vartheta}{dx} = 1 + \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d\vartheta}{dx} - 1$$

Substitute in the equation to get

$$-1 + \frac{d\vartheta}{dx} = \sqrt{\vartheta} - 1 \quad \Rightarrow \quad \frac{d\vartheta}{dx} = \sqrt{\vartheta}$$

$$\frac{d\vartheta}{\sqrt{\vartheta}} = dx \quad \overset{\text{integrating}}{\Longrightarrow} \quad 2\sqrt{\vartheta} = x + c$$

Therefore, $\boxed{2\sqrt{y+x} = x + c}$

2) $\frac{dy}{dx} = \sin(x-y)$ Let $\vartheta = x-y$, $\frac{d\vartheta}{dx} = 1 - \frac{dy}{dx}$

$$1 - \frac{d\vartheta}{dx} = \sin\vartheta \quad \Rightarrow \quad \frac{d\vartheta}{1-\sin\vartheta} = dx$$

$$\Rightarrow \quad \frac{1+\sin\vartheta}{(1-\sin\vartheta)(1+\sin\vartheta)}\, d\vartheta = dx \quad \Rightarrow \quad \frac{1+\sin\vartheta}{\cos^2\vartheta}\, d\vartheta = dx$$

$$\left(\frac{1}{\cos^2\vartheta} + \frac{\sin\vartheta}{\cos^2\vartheta}\right) d\vartheta = dx \quad \overset{\text{integrating}}{\Longrightarrow} \quad \tan\vartheta + \frac{1}{\cos\vartheta} = x + c$$

The solution is $\boxed{\tan(x-y) + \frac{1}{\cos(x-y)} = x + c}$

Equations of the form $(a_1x+b_1y+c_1)dx+(a_2x+b_2y+c_2)dy=0$

1) $(2x-y)\,dx+(4x+y-3)\,dy=0$

Let $x=u+h$ where $2h-k=0$ } so $h=2^{-1}$

$y=v+k$ $\quad 4h+k-3=0$ } $k=1$

so $x=u+\frac{1}{2}$ ; $y=v+1$ , $dx=du$, $dy=dv$

Substitute in the eqn:

$(2u+1-v-1)\,du+(4u+2+v+1-3)\,dv=0$

$$\frac{dv}{du}=\frac{v-2u}{4u+v}=\frac{\frac{v}{u}-2}{4+\frac{v}{u}}$$ ( homogeneous )

Let $z=\frac{v}{u} \Rightarrow \frac{dv}{du}=z+u\frac{dz}{du}$ . Therefore we have

$$z+u\frac{dz}{du}=\frac{z-2}{4+z} \quad ; \quad u\frac{dz}{du}=\frac{z-2}{4+z}-z=\frac{-2-3z-z^2}{4+z}$$

$$\frac{z+4}{z^2+3z+2}\,dz=-\frac{du}{u} \quad ; \quad \left(\frac{3}{z+1}-\frac{2}{z+2}\right)dz=-\frac{du}{u}$$

$$\ln|z+1|^3-\ln|z+2|^2=-\ln|u|+\ln c_1$$

$$\frac{|z+1|^3}{(z+2)^2}=\frac{c_1}{|u|}$$ we can choose

$$\frac{(z+1)^3}{(z+2)^2}=\frac{c}{u} \quad ; \quad \frac{\left(\frac{v}{u}+1\right)^3}{\left(\frac{v}{u}+2\right)^2}=\frac{c}{u} \quad ; \quad \frac{\frac{(v+u)^3}{u^3}}{\frac{(v+2u)^2}{u^2}}=\frac{c}{u}$$

$$(v+u)^3=c\,(v+2u)^2$$

$$\left(y-1+x-\frac{1}{2}\right)^3=c\,(y-1+2x-1)^2$$ or

$$\boxed{\left(y+x-\frac{3}{2}\right)^2=c\,(y+2x-1)^2}$$

2) $(2x+y+4)\,dx + (x-2y+2)\,dy = 0$

$x = u+h$ where $\left.\begin{array}{l} 2h+k+4=0 \\ h-2k+2=0 \end{array}\right\}$ so $h=-2$
$y = v+k$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad k=0$

$x = u-2$ , $y = v$ ; $dx = du$ ; $dy = dv$

$(2u - \cancel{4} + v + \cancel{4})\,dx + (u - \cancel{2} - 2v + \cancel{2})\,dv = 0$

$$\frac{dv}{du} = \frac{v+2u}{2v-u} = \frac{\frac{v}{u}+2}{2\frac{v}{u}-1}$$

Let $z = \dfrac{v}{u}$

$$\frac{dv}{du} = u\frac{dz}{dv} + z$$

$$u\frac{dz}{du} + z = \frac{z+2}{2z-1} \quad , \quad u\frac{dz}{du} = \frac{z+2}{2z-1} - z$$

$$u\frac{dz}{du} = \frac{z+2-2z^2+z}{2z-1} = \frac{2+2z-2z^2}{2z-1}$$

$$\frac{2z-1}{2z^2-2z+2}\,dz = -\frac{du}{u}$$

$$\frac{2z-1}{z^2-z+1}\,dz = -2\frac{du}{u} \qquad \ln(z^2-z+1) = \ln u^{-2} + \ln c$$

$$z^2 - z + 1 = \frac{c}{u^2} \qquad \left(\frac{v}{u}\right)^2 - \left(\frac{v}{u}\right) + 1 = \frac{c}{u^2}$$

so $v^2 - uv + u^2 = c$

$$\boxed{y^2 - (x+2)y + \cancel{v}(x+2)^2 = c}$$