First Order Differential Equations

Separable: $\mathbf{M(x)\,dx = N(y)\,dy}$

Solution: $\int M(x)\,dx = \int N(y)\,dy$

Linear: $\mathbf{y' + p(x)\,y = g(x)}$

Solution: $\mu\, y = \int \mu\, g(x)\,dx$

Integrating Factor: $\mu = e^{\int p(x)\,dx}$

Exact: $\mathbf{M(x,y)\,dx + N(x,y)\,dy = 0}$
where $\frac{\partial}{\partial \mathbf{y}}\mathbf{M\,dy\,dx} = \frac{\partial}{\partial \mathbf{x}}\mathbf{N\,dx\,dy}$

Solution: $\Psi(x,y) = c$ where $\frac{\partial}{\partial x}\Psi = M$, $\frac{\partial}{\partial y}\Psi = N$

$$\Psi = \text{“least common sum”} \begin{cases} \int M(x,y)\,dx \\ \int N(x,y)\,dy \end{cases}$$

To make a non-exact equation become exact:
$\mu\, M(x,y)\,dx + \mu\, N(x,y)\,dy = 0$
Integrating Factor: $\ln \mu = \int \frac{M_y - N_x}{N}\,dx$
or $\ln \mu = \int \frac{N_x - M_y}{M}\,dy$
(*integrals above must be single variable*)

Autonomous: $\mathbf{y' = f(y)}$

$f(y_0) = 0 \Longrightarrow$ equilibrium solution at $y = y_0$
$f(y_0) < 0 \Longrightarrow$ solutions go down at $y = y_0$
$f(y_0) > 0 \Longrightarrow$ solutions go up at $y = y_0$

“unstable equilibrium” = solutions go away
“stable equilibrium” = solutions go towards
“semi-stable equilibrium” = solutions mixed

Homogeneous: $\mathbf{y' = \frac{P(x,y)}{Q(x,y)}}$
P and Q are polynomials in x and y
all $x^n y^m$ have total power $(n+m)$ the same

Multiply: $y' = \frac{P(x,y)}{Q(x,y)} \cdot \frac{\frac{1}{x^{n+m}}}{\frac{1}{x^{n+m}}}$
Substitute: $\left(\frac{y}{x}\right) = v$ and $y' = v + xv'$
(*This converts equation to a separable DE.*)

Bernoulli: $\mathbf{y' + p(x)\,y = q(x)\,y^n}$

Rewrite: $y^{-n}\,y' + p(x)\,y^{1-n} = q(x)$
Substitute: $y^{1-n} = v$ and $y^{-n}\,y' = \frac{1}{1-n}v'$
(*This converts equation to a linear DE.*)

Second Order Differential Equations

Homogeneous Linear, Constant Coefficients:
$\mathbf{a\,y'' + b\,y' + c\,y = 0}$

Characteristic Eqn: $a\,r^2 + b\,r + c = 0$

Solution depends on the type of roots:

- $r = r_1,\ r_2$ (*real, not repeated*)
  $y = c_1\,e^{r_1 x} + c_2\,e^{r_2 x}$
- $r = \alpha \pm \beta\, i$ (*complex conjugates*)
  $y = c_1\,e^{\alpha x}\cos(\beta x) + c_2\,e^{\alpha x}\sin(\beta x)$
- $r = r_0,\ r_0$ (*repeated root*)
  $y = c_1\,e^{r_0 x} + c_2\,x\,e^{r_0 x}$

Reduction of Order:
$\mathbf{y'' + p(x)\,y' + q(x)\,y = 0}$
with one solution $\mathbf{y_1 = y_1(x)}$ known

Substitute: $y = v\,y_1$
$y' = v\,y_1' + v'\,y_1$
$y'' = v\,y_1'' + 2v'\,y_1' + v''\,y_1$

DE becomes: $(2v'\,y_1' + v''\,y_1) + p\,v'\,y_1 = 0$

Separable: $\frac{1}{(v')}\,(v')' = -\left(p + \frac{2\,y_1'}{y_1}\right)$

Undetermined Coefficients:
$\mathbf{y'' + p(x)\,y' + q(x)\,y = g(x)}$
homogeneous solution $\mathbf{y = c_1\,y_1 + c_2\,y_2}$ known

General solution is $y = c_1\,y_1 + c_2\,y_2 + Y_p$
Y_p is a *particular solution*

Find Y_p by guessing a form and then plugging into DE:

- $g = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$
  $Y_p = x^s(\,A_0 x^n + A_1 x^{n-1} + \cdots + A_n\,)$
- $g = (a_0 x^n + a_1 x^{n-1} + \cdots + a_n)\,e^{\alpha x}$
  $Y_p = x^s(\,A_0 x^n + A_1 x^{n-1} + \cdots + A_n\,)\,e^{\alpha x}$
- $g = (a_0 x^n + \cdots + a_n)\,e^{\alpha x}\,\cos(\beta x)$ or $\sin(\beta x)$
  $Y_p = x^s(\,A_0 x^n + \cdots + A_n\,)\,e^{\alpha x}\,\cos(\beta x)$
  $\quad + x^s(\,B_0 x^n + \cdots + B_n\,)\,e^{\alpha x}\,\sin(\beta x)$

(*x^s is chosen so that y_1 and y_2 are not terms of Y_p.*)

Variation of Parameters:
$\mathbf{y'' + p(x)\,y' + q(x)\,y = g(x)}$
homogeneous solution $\mathbf{y = c_1\,y_1 + c_2\,y_2}$ known

General solution is:
$$y = -y_1 \int \frac{y_2\,g}{W(y_1,y_2)}\,dx + y_2 \int \frac{y_1\,g}{W(y_1,y_2)}\,dx$$

Wronskian: $W(y_1,y_2) = y_1\,y_2' - y_1'\,y_2$

Existence and Uniqueness Theorems

First Order, Linear Initial Value Problem:
$\mathbf{y' + p(x)\,y = g(x),\quad y(x_0) = y_0}$

- Solution exists and is unique if p and g are continuous at x_0.
- Solution is defined on the entire interval containing x_0 where p and g are continuous.

Note: higher order linear is the same.

First Order, General Initial Value Problem:
$\mathbf{y' = f(x,y),\quad y(x_0) = y_0}$

- Solution exists if f is continuous at (x_0, y_0).
- It is unique if $\frac{\partial}{\partial y} f$ is continuous at (x_0, y_0).
- Solutions are defined somewhere inside the rectangle containing (x_0, y_0) where f and $\frac{\partial}{\partial y} f$ are continuous.

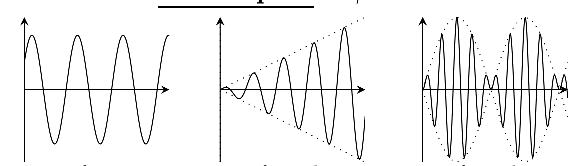
Differential Equations as Vibrations

$$\mathbf{m\,y'' + \gamma\,y' + k\,y = F(x)} \qquad \begin{cases} m & \text{mass} \\ \gamma & \text{dampening} \\ k & \text{spring constant} \\ F & \text{forcing function} \end{cases}$$

- (Undamped) natural freq. $\omega_0 = \sqrt{\frac{k}{m}}$
- (Damped) quasi-frequency $\mu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$

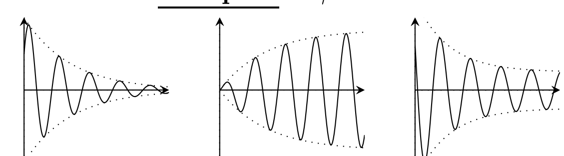
Resonance occurs if forcing freq. $\approx$ system freq.

Undamped: $\gamma = \mathbf{0}$



free: $y = R\cos(\omega_0\,x - \delta)$, $R^2 = c_1^2 + c_2^2$, $\delta = \mathrm{Tan}^{-1}\frac{c_2}{c_1}$ | forced, resonance | forced, no resonance: beat freq $= \frac{1}{2}|\omega_F - \omega_0|$

Damped: $\gamma^2 < \mathbf{4mk}$



free | forced, resonance | forced, no resonance

Not pictured: **overdamped** ($\gamma^2 > 4mk$)
critically damped ($\gamma^2 = 4mk$)