Q1, $\underbrace{(x^3-y)}_{M}dx + \underbrace{x}_{N}dy = 0$; $y(1)=3$.

Solu. $M_y = -1$, $N_x = 1$. $\frac{M_y - N_x}{N} = \frac{-2}{x} = f(x)$.

Hence

$$\frac{d\mu}{\mu} = \frac{-2}{x}dx \qquad \ln|\mu| = -2\ln|x| + \ln c_1 \qquad \mu = \frac{1}{x^2} .$$

So

$$\left(x - \frac{y}{x^2}\right)dx + \frac{1}{x}dy = 0 \text{ . (exact).}$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= x - \frac{y}{x^2} \\ \frac{\partial F}{\partial y} &= \frac{1}{x} \end{aligned}\right\} \Rightarrow F(x,y) = \frac{y}{x} + \varphi(x)$$

$$\cancel{-\frac{y}{x^2}} + \frac{d\varphi}{dx} = x - \cancel{\frac{y}{x^2}} \qquad \varphi(x) = \frac{x^2}{2} + c_1$$

The solution is $\frac{y}{x} + \frac{x^2}{2} = c$. $\frac{3}{1} + \frac{1}{2} = c \Rightarrow c = \frac{7}{2}$

and $\boxed{\frac{y}{x} + \frac{x^2}{2} = \frac{7}{2}}$ or $\boxed{y = \frac{7x}{2} - \frac{x^3}{2}}$.

II $\frac{dy}{dx} = \frac{y - x^3}{x}$ $\qquad \frac{dy}{dx} \underbrace{-\frac{1}{x}}_{P(x)} y = \underbrace{-x^2}_{Q(x)}$ (linear)

$$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} \Rightarrow \mu(x) = \frac{1}{x}$$

$$\frac{1}{x}y = -\int \frac{1}{x}x^2 dx \qquad \frac{1}{x}y = -\frac{x^2}{2} + c \qquad y = -\frac{x^3}{2} + cx .$$

$$3 = -\frac{1}{2} + c \qquad c = \frac{7}{2} \Rightarrow \boxed{y = -\frac{x^3}{2} + \frac{7x}{2}}$$

Q1. $\underbrace{(x^3-y)}_{M}dx + \underbrace{x}_{N}dy = 0 \quad ; \quad y(1) = 3.$

Soln. $M_y = -1, \quad N_x = 1. \qquad \dfrac{M_y - N_x}{N} = \dfrac{-2}{x} = f(x).$

Hence

$$\frac{d\mu}{\mu} = \frac{-2}{x}dx \qquad \ln|\mu| = -2\ln|x| + \ln c_1 \qquad \mu = \frac{1}{x^2}.$$

So

$$\left(x - \frac{y}{x^2}\right)dx + \frac{1}{x}dy = 0. \text{ (exact)}.$$

$$\frac{\partial F}{\partial x} = x - \frac{y}{x^2} \qquad F(x,y) = \frac{y}{x} + \varphi(x)$$

$$\frac{\partial F}{\partial y} = \frac{1}{x} \qquad -\cancel{\frac{y}{x^2}} + \frac{d\varphi}{dx} = x - \cancel{\frac{y}{x^2}} \qquad \varphi(x) = \frac{x^2}{2} + c_1$$

The solution is $\dfrac{y}{x} + \dfrac{x^2}{2} = c. \qquad \dfrac{3}{1} + \dfrac{1}{2} = c \Rightarrow c = \dfrac{7}{2}$

and $\boxed{\dfrac{y}{x} + \dfrac{x^2}{2} = \dfrac{7}{2}}$ or $\boxed{y = \dfrac{7x}{2} - \dfrac{x^3}{2}}$.

II $\dfrac{dy}{dx} = \dfrac{y - x^3}{x} \qquad \dfrac{dy}{dx} \underbrace{-\dfrac{1}{x}}_{P(x)} y = \underbrace{-x^2}_{Q(x)}$ (linear)

$$\mu(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} \Rightarrow \mu(x) = \frac{1}{x}$$

$$\frac{1}{x}y = -\int \frac{1}{x}x^2 dx \qquad \frac{1}{x}y = -\frac{x^2}{2} + c \qquad y = -\frac{x^3}{2} + cx.$$

$$3 = -\frac{1}{2} + c \qquad c = \frac{7}{2} \Rightarrow \boxed{y = -\frac{x^3}{2} + \frac{7x}{2}}$$

Ex. $\underbrace{(2x^2+y)}_{M}\,dx + \underbrace{(x^2y-x)}_{N}\,dy = 0.$

Solu. $M_y = 1$, $N_x = 2xy-1$. (not exact).

$$\frac{M_y - N_x}{N} = \frac{1-2xy+1}{x(xy-1)} = \frac{2(1-xy)}{x(xy-1)} = \frac{-2}{x} = f(x)$$

Thus

$$\frac{d\mu}{\mu} = \frac{-2}{x}\,dx \qquad \ln|\mu| = -2\ln|x| + \ln c_1 \qquad \mu(x) = \frac{1}{x^2}$$

Multiplying by $\frac{1}{x^2}$ we have

$$\underbrace{\left(2+\frac{y}{x^2}\right)}_{\tilde{M}}dx + \underbrace{\left(y-\frac{1}{x}\right)}_{\tilde{N}}dy = 0 \quad \text{. (exact)}$$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2+\frac{y}{x^2} \\ \frac{\partial F}{\partial y} &= y-\frac{1}{x} \end{aligned}\right\} \qquad \begin{aligned} &F(x,y) = 2x - \frac{y}{x} + \varphi(y) \\ &-\frac{1}{x} + \frac{d\varphi}{dy} = y - \frac{1}{x} \Rightarrow \varphi(y) = \frac{y^2}{2} + c_1 \end{aligned}$$

Hence the solution is $\boxed{2x - \frac{y}{x} + \frac{y^2}{2} = C}$

Ex. Solve the eq. $x(4y\,dx + 2x\,dy) + y^3(3y\,dx + 5x\,dy) = 0$.

Soln. Since $\underset{a_1 b_2 - a_2 b_1}{4.5 - 2.3 \neq 0}$ there is an integrating factor of the form $x^m y^n$.

Hence

$$\left(4x^{m+1}y^{n+1}\,dx + 2x^{m+2}y^{n}\,dy\right) + \left(3x^{m}y^{n+4}\,dx + 5x^{m+1}y^{n+3}\,dy\right) = 0$$

$$\underbrace{\left(4x^{m+1}y^{n+1} + 3x^{m}y^{n+4}\right)}_{M}dx + \underbrace{\left(2x^{m+2}y^{n} + 5x^{m+1}y^{n+3}\right)}_{N}dy = 0$$

$$M_y = 4(n+1)x^{m+1}y^{n} + 3(n+4)x^{m}y^{n+3}$$

$$N_x = 2(m+2)x^{m+1}y^{n} + 5(m+1)x^{m}y^{n+3}$$

$4(n+1) = 2(m+2) \Rightarrow 2n+\cancel{2} = m+\cancel{2}$

$3(n+4) = 5(m+1) \Rightarrow 3n+12 = 5m+5$

$7n = 7 \Rightarrow n = 1$ and $m = 2$

Hence an integrating factor is $\mu(x,y) = x^2 y$. Then

$$\left(4x^3y^2 + 3x^2y^5\right)dx + \left(2x^4y + 5x^3y^4\right)dy = 0$$

is exact.

$$\frac{\partial F}{\partial x} = 4x^3y^2 + 3x^2y^5 \qquad F(x,y) = x^4y^2 + x^3y^5 + \varphi(y)$$

$$\frac{\partial F}{\partial y} = 2x^4y + 5x^3y^4 \qquad 2x^4y + 5x^3y^4 + \frac{d\varphi}{dy} = 2x^4y + 5x^3y^4 \Rightarrow \varphi(y) = c_1$$

$$\boxed{x^4y^2 + x^3y^5 = c}$$