ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 258 - Introduction to Differential Equations

Final EXAMINATION
30.05.2017

STUDENT NUMBER:

NAME-SURNAME: ANSWER KEY

SIGNATURE:

INSTRUCTOR:

DURATION: 110 minutes

Question	Grade	Out of
1		10
2		20
3		20
4		20
5		20
6		20
Total		110

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.

2) Check that the exam paper contains 6 problems.

3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Solve the following integro-differential equation

$$y' = \sin x + \int_0^x y(x-u)\cos u du, \qquad y(0) = 0.$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{\sin x + y(x) * \cos x\}$$

$$sY(s) - \cancel{y(0)}^{0} = \mathcal{L}\{\sin x\} + \mathcal{L}\{y\}\mathcal{L}\{\cos x\}$$

$$sY(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1}Y(s)$$

$$\left(s - \frac{s}{s^2+1}\right)Y(s) = \frac{1}{s^2+1}$$

$$\frac{s^3+s-s}{s^2+1}Y(s) = \frac{1}{s^2+1} \Rightarrow Y(s) = \frac{1}{s^3}$$

$$\boxed{y(x) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{x^2}{2}}$$

Question 2. Use Laplace transforms to solve the following IVP

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 < x < 2, \\ 0, & x > 2, \end{cases} \qquad y(0) = y'(0) = 0.$$

$$y'' + 2y' + 5y = 1 - u(x-2)$$

$$\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{1 - u(x-2)\} \Rightarrow$$

$$s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)} + 2[sY(s) - \cancel{y(0)}] + 5Y(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$(s^2 + 2s + 5)Y(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

$$Y(s) = \frac{1}{s(s^2+2s+5)} - \frac{e^{-2s}}{s(s^2+2s+5)}$$

$$\frac{1}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} = \frac{(A+B)s^2 + (2A+C)s + 5A}{s(s^2+2s+5)}$$

$A + B = 0$, $2A + C = 0$; $5A = 1$ so $A = \frac{1}{5}$ $B = -\frac{1}{5}$ $C = -\frac{2}{5}$

$$\frac{1}{s(s^2+2s+5)} = \frac{\frac{1}{5}}{s} + \frac{-\frac{1}{5}s - \frac{2}{5}}{s^2+2s+5} = \frac{\frac{1}{5}}{s} - \frac{1}{5}\frac{s+2}{s^2+2s+5} = \frac{\frac{1}{5}}{s} - \frac{1}{5}\frac{s+1}{(s+1)^2+4} - \frac{1}{10}\frac{2}{(s+1)^2+4}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+5)}\right\} = \frac{1}{5} - \frac{1}{5}e^{-x}\cos 2x - \frac{1}{10}e^{-x}\sin 2x$$

$$y(x) = \frac{1}{5} - \frac{1}{5}e^{-x}\cos 2x - \frac{1}{10}e^{-x}\sin 2x - \left[\frac{1}{5} - \frac{1}{5}e^{-x+2}\cos(2x-4) - \frac{1}{10}e^{-x+2}\sin(2x-4)\right]u(x-$$

Question 3. Using power series around $x = 0$, find the general solution of

$$(1+x^2)y'' - 4xy' + 6y = 0.$$

$p(x) = \frac{-4x}{1+x^2}$, $q(x) = \frac{6}{1+x^2}$ are analytic at $x=0$. So $x=0$ is an ordinary point.

$$y = \sum_{n=0}^{\infty} a_n x^n \; ; \; y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \; ; \; y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Thus we have

$$(1+x^2) \sum_{n=2}^{\infty} n(n-1) x^{n-2} a_n - 4x \sum_{n=1}^{\infty} n a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n - 4 \sum_{n=1}^{\infty} n a_n x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^n - 4 \sum_{n=1}^{\infty} n a_n x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2a_2 + 6a_0) + (6a_3 - 4a_1 + 6a_1)x + \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + n(n-1) a_n - 4n a_n + 6 a_n \right] x^n = 0$$

$$2a_2 + 6a_0 = 0 \implies a_2 = -3a_0$$

$$6a_3 + 2a_1 = 0 \implies a_3 = -\frac{1}{3} a_1$$

$$(n+2)(n+1) a_{n+2} + (n^2 - 5n + 6) a_n = 0 \qquad n \geq 2$$

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)} a_n \qquad n \geq 0$$

$$\left.\begin{array}{l} n=2 \implies a_4 = 0 \\ n=5 \implies a_5 = 0 \end{array}\right\} \implies a_n = 0 \quad n \geq 4$$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + 0$$

$$= a_0 + a_1 x - 3a_0 x^2 - \frac{1}{3} a_1 x^3$$

$$\boxed{y = a_0 (1 - 3x^2) + a_1 \left(x - \frac{1}{3} x^3\right)}$$

Question 4. Find the general solution of $x^2y'' + xy' + y = \csc(\ln x)$.

This is a Cauchy-Euler eqn.

Let $x = e^t \Rightarrow t = \ln x$

$$x^2 \cdot \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + x \cdot \frac{1}{x}\frac{dy}{dt} + y = \csc t$$

$$\frac{d^2y}{dt^2} + y = \csc t \qquad y_h = c_1 \cos t + c_2 \sin t$$

$$\lambda^2 + 1 = 0 \quad \lambda = \pm i \qquad y_p = v_1 \cos t + v_2 \sin t$$

$$v_1' \cos t + v_2' \sin t = 0$$

$$v_1'(-\sin t) + v_2' \cos t = \csc t$$

$$v_1' = \frac{\begin{vmatrix} 0 & \sin t \\ \csc t & \cos t \end{vmatrix}}{\begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}} = -\frac{\sin t \csc t}{\cos^2 t + \sin^2 t} = -1. \Rightarrow v_1 = -t$$

$$v_2' = \frac{\begin{vmatrix} \cos t & 0 \\ -\sin t & \csc t \end{vmatrix}}{1} = \frac{\cos t}{\sin t} \Rightarrow v_2 = \ln|\sin t|$$

$$y_p = -t\cos t + \sin t \ln|\sin t|$$

$$y = y_h + y_p = c_1 \cos t + c_2 \sin t - t\cos t + \sin t \ln|\sin t|$$

$$\boxed{y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x) - \ln x \cos(\ln x) + \sin(\ln x)\ln|\sin \ln x|}$$

Question 5. Solve the IVP

$$y'' - 3y' + 2y = -2xe^x, \qquad y(0) = 0,\ y'(0) = 1.$$

$\lambda^2 - 3\lambda + 2 = 0 \implies \lambda = 1, 2 \qquad y_h = c_1 e^x + c_2 e^{2x}$

$y_p = x^s (Ax + B) e^x \qquad s = 1$

$y_p = (Ax^2 + Bx) e^x$

$y_p' = (2Ax + B) e^x + (Ax^2 + Bx) e^x = [Ax^2 + (2A+B)x + B] e^x$

$y_p'' = [2Ax + (2A+B)] e^x + [Ax^2 + (2A+B)x + B] e^x$

$= [Ax^2 + (4A+B)x + 2A + 2B] e^x$

Plug in the nonhomogeneous equations to get.

$[\cancel{Ax^2} + (4A + \cancel{B})x + 2A + 2B] e^x - 3[\cancel{Ax^2} + (2A + \cancel{B})x + B] e^x + 2(\cancel{Ax^2} + \cancel{Bx}) e^x = -2xe^x$

$[(4A - 6A)x + 2A - B] e^x = -2xe^x$

$-2A = -2 \qquad A = 1 \qquad 2A - B = 0 \implies B = 2$

$y_p = (x^2 + 2x) e^x \implies y = c_1 e^x + c_2 e^{2x} + (x^2 + 2x) e^x$

$y(0) = 0 \implies c_1 + c_2 + 0 = 0 \qquad \implies c_2 = -1$

$y'(0) = 1 \implies c_1 + 2c_2 + 2 = 1 \qquad c_1 = 1$

$$\boxed{y = e^x - e^{2x} + (x^2 + 2x) e^x}$$

Question 6. Solve the following partial differential equation

$$\frac{\partial u}{\partial t} = 10\,\frac{\partial^2 u}{\partial x^2}\,, \quad 0 < x < \pi,\ t > 0,$$

$$u(0,t) = 0, \quad u(\pi,t) = \pi, \qquad t > 0,$$

$$u(x,0) = \begin{cases} 0, & 0 < x < \frac{\pi}{2}, \\ \pi, & \frac{\pi}{2} < x < \pi. \end{cases}$$

$$u(x,t) = \frac{\pi - 0}{\pi}x + \sum_{n=1}^{\infty} c_n \sin nx\, e^{-10n^2 t}$$

$$= x + \sum_{n=1}^{\infty} c_n \sin nx\, e^{-10n^2 t}$$

$$c_n = \frac{2}{\pi}\int_0^{\pi} \left(\begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ \pi & \frac{\pi}{2} < x < \pi \end{cases} - x \right) \cdot \sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\begin{cases} -x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \pi \end{cases} \right) \cdot \sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} -x \sin nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\left(\frac{x\cos nx}{n} - \frac{\sin nx}{n^2} \right|_0^{\pi/2} + \frac{2}{\pi}\left(\frac{(x-\pi)\cos nx}{n} - \frac{\sin nx}{n^2} \right|_{\pi/2}^{\pi} \right.$$

$$= \frac{2}{\pi}\left[\frac{\pi \cos\frac{n}{2}\pi}{2n} - \cancel{\frac{\sin\frac{n\pi}{2}}{n^2}} + \frac{\pi}{2n}\cos\frac{n}{2}\pi + \cancel{\frac{\sin\frac{n\pi}{2}}{n^2}} \right]$$

$$= 2\cos\frac{n}{2}\pi$$

$$\boxed{u(x,t) = x + \sum_{n=1}^{\infty} 2\cos\frac{n\pi}{2}\sin nx\, e^{-10n^2 t}}$$