



ÇANKAYA UNIVERSITY

Department of Mathematics

MATH 258 - Introduction to Differential Equations

SECOND MIDTERM EXAMINATION

24.04.2017

STUDENT NUMBER:

NAME-SURNAME:

SIGNATURE:

INSTRUCTOR:

DURATION: 110 minutes

Answer key

Question	Grade	Out of
1		12
2		20
3		20
4		20
5		13
6		20
Total		105

IMPORTANT NOTES:

- 1) Please make sure that you have written your student number and name above.
- 2) Check that the exam paper contains 6 problems.
- 3) Show all your work. No points will be given to correct answers without reasonable work.

Question 1. Given that $y_1 = e^x$ is a solution of the equation $xy'' - (x+1)y' + y = 0$. Find a second linearly independent solution of the given equation.

$$y_2 = e^{x\vartheta} \quad y'_2 = e^{x\vartheta} + e^{x\vartheta'}; \quad y''_2 = e^{x\vartheta} + 2e^{x\vartheta'} + e^{x\vartheta''}$$

Putting these in the eqn, we get

$$\cancel{xe^{x\vartheta} + 2xe^{x\vartheta'} + xe^{x\vartheta''}} - \cancel{xe^{x\vartheta}} - \cancel{xe^{x\vartheta'}} - \cancel{e^{x\vartheta} - e^{x\vartheta'}} + \cancel{e^{x\vartheta}} = 0$$

$$xe^{x\vartheta''} + (x-1)e^{x\vartheta'} = 0$$

$$\vartheta'' + \frac{x-1}{x}\vartheta' = 0 \quad \text{Let } w = \vartheta'$$

$$w' = -\frac{x-1}{x}w$$

$$\frac{dw}{w} = \left(1 + \frac{1}{x}\right) dx$$

$$\ln w = -x + \ln x \Rightarrow w = x e^{-x}$$

$$\vartheta' = x e^{-x}; \quad \vartheta = \int x e^{-x} dx = -(x+1) e^{-x}$$

$$\text{so } y_2 = e^x \left[-(x+1) e^{-x} \right] = -(x+1) \quad \text{or } y_2 = x+1$$

Question 2. Find the general solution of $y''' - y'' = 6 + e^x$.

$$r^3 - r^2 = 0 \quad r^2(r-1) = 0 \quad r=0, r=1$$

$$y_h = C_1 + C_2 x + C_3 e^x$$

$$y_p = Ax^2 + Bx e^x$$

$$y_p' = 2Ax + Be^x + Bxe^x$$

$$y_p'' = 2A + 2Be^x + Bxe^x$$

$$y_p''' = 3Be^x + Bxe^x$$

$$\cancel{3Be^x} + \cancel{Bxe^x} - 2A - 2Be^x - Bxe^x = 6 + e^x$$

$$-2A = 6 \Rightarrow A = -3$$

$$B = 1$$

$$y = y_h + y_p = C_1 + C_2 x + C_3 e^x - 3x^2 + x e^x$$

Question 3. Find the general solution of $y'' - 2y' + y = e^x \ln x$.

$$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$$

$$y_h = c_1 e^x + c_2 x e^x$$

$$y_p = v_1 e^x + v_2 x e^x$$

$$e^x v_1' + v_2' x e^x = 0$$

$$v_2' e^x + v_2' (x+1) e^x = e^x \ln x$$

$$v_2' = \frac{\begin{vmatrix} 0 & x e^x \\ e^x \ln x & (x+1) e^x \end{vmatrix}}{\begin{vmatrix} e^x & x e^x \\ e^x & (x+1) e^x \end{vmatrix}} = \frac{-x e^{2x} \ln x}{e^{2x}} = -x \ln x$$

$$v_1 = \int -x \ln x dx = -\frac{x^2}{2} \ln x + \frac{x^2}{4}$$

$$v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x \ln x \end{vmatrix}}{e^{2x}} = \frac{e^{2x} \ln x}{e^{2x}} = \ln x$$

$$v_2 = \int \ln x dx = x \ln x - x$$

$$\begin{aligned} y_p &= -e^x \frac{x^2}{2} \ln x + \frac{x^2}{4} e^x + x^2 e^x \ln x - x^2 e^x \\ &= \frac{1}{2} x^2 e^x \ln x - \frac{3}{4} x^2 e^x \end{aligned}$$

$$y = y_h + y_p = c_1 e^x + c_2 x e^x + \frac{1}{2} x^2 e^x \ln x - \frac{3}{4} x^2 e^x$$

Question 4. Find the general solution of $x^2y'' - xy' + y = 4x$, $x > 0$.

This is a Cauchy-Euler Equation

Set

$$x = e^t ; t = \ln x$$

$$x^2 \cdot \frac{1}{x^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt} \right) - x \cdot \frac{1}{x} \frac{dy}{dt} + y = 4e^t$$

$$\frac{d^2y}{dt^2} - 2 \frac{dy}{dt} + y = 4e^t$$

$$y_n = c_1 e^t + c_2 t e^t$$

$$y_p = At^2 e^t$$

$$\frac{dy_p}{dt} = 2At e^t + At^2 e^t$$

$$\frac{d^2y_p}{dt^2} = 2Ae^t + 4At e^t + At^2 e^t$$

$$\cancel{2Ae^t + 4At e^t + At^2 e^t} - 2(2At e^t + At^2 e^t) + At^2 e^t = 4e^t$$

$$2A = 4 \quad A = 2 \quad y_p = 2t^2 e^t$$

$$y = y_n + y_p = c_1 e^t + c_2 t e^t + 2t^2 e^t$$

$$y(x) = c_1 x + c_2 x \ln x + 2x \ln^2 x$$

Question 5. Find the constants a , b and c if e^{2x} and e^{-3x} are solutions of the homogeneous equation $y'' + ay' + by = 0$ and $x \sin x$ is a particular solution of $y'' + ay' + by = (x+2) \cos x + (1-cx) \sin x$.

$$r=2 \\ r=3$$

$$(r-2)(r+3)=0 \\ r^2+r-6=0 \\ r^2+ar+b=0$$

$$\boxed{a=1} \\ b=-6$$

we have now $y'' + y' - 6y = (x+2) \cos x + (1-cx) \sin x$

$$y_p = x \sin x \\ y'_p = \sin x + x \cos x \\ y''_p = 2 \cos x - x \sin x$$

$$2 \cos x - x \sin x + \sin x + x \cos x - 6x \sin x = (x+2) \cos x + (1-7x) \sin x \\ = (x+2) \cos x + (1-cx) \sin x$$

$$\boxed{c=7}$$

Question 6. Using Laplace transforms, solve the IVP

$$y'' - 2y' + y = 4 \sin x \cos x e^x; \quad y(0) = 0, \quad y'(0) = 1.$$

$$\mathcal{L}\{y'' - 2y' + y\} = \mathcal{L}\{4 \sin x \cos x e^x\} = \mathcal{L}\{2 \sin 2x e^x\}$$

$$s^2 Y(s) - s y(0) - y'(0) - 2[sY(s) - y(0)] + Y(s) = \frac{-4}{(s-1)^2 + 4}$$

$$(s-1)^2 Y(s) = \frac{4}{(s-1)^2 + 4}$$

$$Y(s) = \frac{4}{(s-1)^2 [(s-1)^2 + 4]}$$

$$= \frac{1}{(s-1)^2} - \frac{1}{(s-1)^2 + 4}$$

$$y(x) = \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2} - \frac{1}{(s-1)^2 + 4} \right\}$$

$$= x e^x - \frac{1}{2} e^x \sin 2x$$