

1) The roots of the characteristic polynomial of the homogenous equation associated with

$$y''' + ay'' + by' + cy = 2e^x,$$

are 0, 1, and -1, where $a, b, c \in \mathbb{R}$.

a) Find a, b, c .

b) Find the general solution of the given equation.

a) Char eqn $r^3 + ar^2 + br + c = 0$ 2

$$r=0 \Rightarrow \boxed{c=0} \quad 2$$

$$r=1 \Rightarrow 1+a+b=0 \quad \left. \begin{array}{l} \boxed{b=-1} \\ \boxed{a=0} \end{array} \right. \quad 2$$

$$r=-1 \Rightarrow -1+a-b=0$$

8

b) The eqn is

4 } $y''' - y' = 2e^x$
 $y_h = C_1 + C_2 e^x + C_3 e^{-x}$

12

$$y_p = Ax e^x$$

$$y_p' = A(x+1)e^x$$

$$y_p'' = A(x+2)e^x$$

$$y_p''' = A(x+3)e^x$$

Substitute

6 } $A(x+3)e^x - A(x+1)e^x = 2e^x$
 $2Ae^x = 2e^x \Rightarrow \boxed{A=1}$

$$\text{so, } y_p = x e^x$$

The general soln is

$$y = y_h + y_p$$

$$= C_1 + C_2 e^x + C_3 e^{-x} + x e^x$$

2

2) Consider the equation,

$$xy'' + xy' + y = \sec(\ln x). \quad (1)$$

a) Show that the substitution $x = e^t$ will put the given equation in the form

$$\frac{d^2y}{dt^2} + y = \sec t. \quad (2)$$

b) Solve the transformed Equation (2).

c) Find the general solution of the equation given by Equation (1).

a) $x = e^t$ and $t = \ln x$

$$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt} \right)$$

$$x^2 \frac{1}{x^2} \left(\frac{d^2y}{dt^2} - \frac{dy}{dt} \right) + x \frac{1}{x} \frac{dy}{dt} + y = \sec t$$

b) $\frac{d^2y}{dt^2} + y = \sec t$

$$r^2 + 1 = 0 \quad r = \pm i$$

$$y_h = c_1 \cos t + c_2 \sin t$$

$$y_p = -\cos t \int \frac{\sin t \sec t}{1+1} dt + \sin t \int \frac{\cos t \sec t}{1+1} dt$$

$$= \cos t \ln |\cos t| + t \sin t$$

$$y = y_h + y_p = c_1 \cos t + c_2 \sin t + \cos t \ln |\cos t| + t \sin t$$

c) $y = c_1 \cos(\ln x) + c_2 \sin(\ln x) +$
 $\cos(\ln x) \ln |\cos(\ln x)| + \ln x \sin(\ln x)$

OR y_p is calculated as

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

$$G_1 = \frac{1}{1} \begin{vmatrix} \sin t & \cos t \\ \cos t & \cos t \end{vmatrix} = -\sin t \sec t$$

$$G_2 = \frac{1}{1} \begin{vmatrix} \cos t & \sin t \\ \sin t & \cos t \end{vmatrix} = \cos t \sec t$$

3) Use Laplace transforms to solve the IVP

$$y'' + y = \cos t, \quad y(0) = y'(0) = 0.$$

$$\mathcal{L}\{y'' + y\} = \mathcal{L}\{\cos t\}$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{s}{s^2 + 1}$$

$$Y(s) = \frac{s}{(s^2 + 1)^2} = \frac{s}{s^2 + 1} \cdot \frac{1}{s^2 + 1}$$

$$y = \mathcal{F}^{-1}\left\{ \frac{1}{s^2 + 1} \cdot \frac{s}{s^2 + 1} \right\} = \sin t + \cos t$$

$$= \int_0^t \sin \tau \cos(t-\tau) d\tau$$

$$= \frac{1}{2} \int_0^t (\sin t + \sin(2\tau - t)) d\tau$$

$$= \frac{1}{2} \left[t \sin t - \frac{1}{2} \cos(2t - t) \right] \Big|_0^t$$

$$= \frac{1}{2} \left[t \sin t - \frac{1}{2} \cos t - 0 + \frac{1}{2} \cos(-t) \right]$$

$$= \frac{1}{2} t \sin t$$

4) Use Laplace transforms to solve the IVP

$$y'' + 4y' + 4y = \underbrace{\begin{cases} 0 & 0 < t < 1, \\ 4 & t > 1, \end{cases}}_{4u(t-1)}, \quad y(0) = y'(0) = 0.$$

$$s^2 Y(s) - s y(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = 4 \frac{e^{-s}}{s}$$

$$(s^2 + 4s + 4)Y(s) = \frac{4e^{-s}}{s}$$

$$Y(s) = \frac{4e^{-s}}{s(s+2)^2}$$

$$\frac{4}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{A(s+2)^2 + Bs(s+2) + Cs}{s(s+2)^2}$$

$$4 = A(s+2)^2 + Bs(s+2) + Cs$$

$$s=0 \Rightarrow A=1$$

$$s=-2 \Rightarrow C=-2$$

$$s=1 \Rightarrow B=-1$$

$$\frac{4}{s(s+2)^2} = \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}$$

$$f^{-1} \left\{ \frac{4}{s(s+2)^2} \right\} = 1 - e^{-2t} - 2t e^{-2t}$$

$$y = \left(1 - e^{-2(t-1)} - 2(t-1)e^{-2(t-1)} \right) u(t-1)$$

5) Find the power series solutions of the IVP

$$(1+x^2)y'' + 4xy' - 10y = 0, \quad y(0) = 1; \quad y'(0) = 0.$$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad a_0 = 1 \quad a_1 = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n + 4 \sum_{n=1}^{\infty} n a_n x^n - 10 \sum_{n=0}^{\infty} a_n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} (n^2 + 3n - 10)a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n^2 + 3n - 10)a_n] x^n = 0$$

$$a_{n+2} = -\frac{n^2 + 3n - 10}{(n+2)(n+1)} a_n \quad n \geq 0$$

$$n=0 \quad a_2 = +\frac{10}{2} a_0 = 5$$

$$n=1 \quad a_3 = (-) a_1 = 0 \Rightarrow a_n = 0 \quad n \geq 3$$

$$n=2 \quad a_4 = 0$$

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 \\ &= 1 + 5x^2 \end{aligned}$$

6) Let

$$f(x) = \begin{cases} x & -\pi < x < -\frac{\pi}{2}, \\ 0 & -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ x & \frac{\pi}{2} < x < \pi. \end{cases} \quad f(x+2\pi) = f(x), \forall x.$$

Find the Fourier series of $f(x)$ and determine the function $g(x)$ to which this series converges.

f is an odd func $a_0 = a_n = 0$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi x \sin nx dx = \frac{2}{\pi} \int_{\pi/2}^\pi x \sin nx dx \\ &= \frac{2}{\pi} \left(-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right) \Big|_{\pi/2}^\pi \\ &= \frac{2}{\pi} \left(-\frac{\pi}{n} (-1)^n + \frac{1}{n^2} \cos \frac{n\pi}{2} - \frac{1}{n^2} \sin \frac{n\pi}{2} \right) \sin nx \end{aligned}$$

$$g(x) = \begin{cases} 0 & x = -\pi \\ x & -\pi < x < -\pi/2 \\ -\pi/4 & x = -\pi/2 \\ 0 & -\pi/2 < x < \pi/2 \\ \pi/4 & x = \pi/2 \\ x & \pi/2 < x < \pi \\ 0 & x = \pi \end{cases}$$