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1) Find the general solution of

$$y''' - 5y'' + 6y' = 6 + 10e^x.$$

$$r^3 - 5r^2 + 6r = 0 \quad r=0, 2, 3$$

$$y_h = c_1 + c_2 e^{2x} + c_3 e^{3x}$$

$$y_p = Ax + Be^x$$

$$y_p' = A + Be^x$$

$$y_p'' = Be^x$$

$$y_p''' = Be^x$$

$$Be^x - 5Be^x + 6A + 6Be^x = 6 + 10e^x \quad \begin{matrix} A=1 \\ B=5 \end{matrix}$$

$$y_p = x + 5e^x$$

$$y = y_h + y_p = c_1 + c_2 e^{2x} + c_3 e^{3x} + x + 5e^x$$

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2) Consider the equation,

$$xy'' - (1+x)y' + y = x^2 e^{2x}, \quad x > 0.$$

a) Verify that $y_1 = x+1$ and $y_2 = e^x$ are two linearly independent solutions of the homogenous equation associated with the given equation.

b) Find the general solution of the given equation.

a) $y_1 = x+1, y_2 = e^x$
 $y_1' = 1, y_2' = e^x$

$$\begin{aligned} x \cdot 0 - (1+x) \cdot 1 + (x+1) &= 0 \\ x e^x - (1+x) e^x + e^x &= e^x(x-1-x+1) \\ &= 0 \end{aligned}$$

$$W(y_1, y_2) = \begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix} = x e^x > 0 \quad \text{so } y_1 \text{ & } y_2 \text{ are lin. indep.}$$

b) $y_p = -(x+1) \int \frac{e^x x^2 e^{2x}}{x \cdot x e^x} dx + e^x \int (x+1) \frac{x^2 e^{2x}}{x \cdot x e^x} dx$
 $= -(x+1) \cdot \int e^{2x} dx + e^x \int (x+1) e^x dx$
 $= -\frac{(x+1)}{2} e^{2x} + x e^{2x} = \left(\frac{x-1}{2}\right) e^{2x}$
 $y = y_h + y_p = c_1(x+1) + c_2 e^x + \left(\frac{x-1}{2}\right) e^{2x}$

OR $y_p = \psi_1 y_1 + \psi_2 y_2$ and, $\begin{aligned} \psi_1' y_1 + \psi_2' y_2 &= 0 \\ \psi_1' y_1' + \psi_2' y_2' &= x^2 e^{2x} \end{aligned}$

$$\begin{bmatrix} x+1 & e^x \\ 1 & e^x \end{bmatrix} \begin{bmatrix} \psi_1' \\ \psi_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x^2 e^{2x} \end{bmatrix}$$

$$\psi_1' = \frac{\begin{vmatrix} 0 & e^x \\ x e^{2x} & e^x \end{vmatrix}}{\begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix}} = \frac{-x^2 e^{3x}}{e^x \cdot x} = -e^{2x} \Rightarrow \psi_1 = -\frac{e^{2x}}{2}$$

$$\psi_2' = \frac{\begin{vmatrix} x+1 & 0 \\ 1 & x e^{2x} \end{vmatrix}}{\begin{vmatrix} x+1 & e^x \\ 1 & e^x \end{vmatrix}} = \frac{x e^{2x} (x+1)}{e^x \cdot x} = (x+1) e^x \Rightarrow \psi_2 = (x+1) e^x - e^x$$

$$\begin{aligned} y_p &= (x+1) \left(-\frac{e^{2x}}{2}\right) + ((x+1) e^x - e^x) e^x \\ &= \left(\frac{x-1}{2}\right) e^{2x} \end{aligned}$$

3) Find the general solution of

$$(3x+1)^2 y'' - 3(3x+1)y' + 9y = 18(3x+1), \quad x > -\frac{1}{3}.$$

$$3x+1 = e^t \quad t = \ln(3x+1)$$

$$y' = \frac{dy}{dt} = \frac{3}{3x+1}$$

$$y'' = \frac{d^2y}{dt^2} = \frac{9}{(3x+1)^2} = \frac{9}{(3x+1)^2} \cdot \frac{dy}{dt}$$

$$9\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 9\frac{dy}{dt} + 9y = 18e^t$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 2e^t$$

$$y_h = c_1 e^t + c_2 t e^t$$

$$y_p = A t^2 e^t$$

$$\frac{dy_p}{dt} = 2Ate^t + At^2e^t, \quad \frac{d^2y_p}{dt^2} = 2Ae^t + 4At^2e^t + At^2e^t$$

$$2Ae^t + 4At^2e^t + At^2e^t - 4At^2e^t - 2At^2e^t + At^2e^t = 2e^t$$

$$A=1$$

$$y_p = t^2 e^t$$

$$y = c_1 e^t + c_2 t e^t + t^2 e^t$$

$$y = c_1 (3x+1) + c_2 (3x+1) \ln(3x+1) + (3x+1) \ln^2(3x+1)$$

- 4) Find the following
 a) $\mathcal{L}^{-1}\{3t^2 - te^{2t} \sin 3t\}$

$$3t^2 \{t^2\} - t^2 \{t e^{2t} \sin 3t\}$$

$$= 3 \frac{2!}{s^3} + \frac{6(s-2)}{(s-2)^2 + 9}$$

b) $\mathcal{L}^{-1}\left\{\frac{s^3 + 4s - 4}{s^4 + 4s^2}\right\}$

$$\frac{s^3 + 4s - 4}{s^2(s^2 + 4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 4}$$

$$s^3 + 4s - 4 = As^3 + 4As + Bs^2 + Cs^2 + Ds$$

$$A + C = 1 \Rightarrow C = 0$$

$$B + D = 0 \Rightarrow D = -B$$

$$4A = 1 \Rightarrow A = \frac{1}{4}$$

$$4B = -4 \Rightarrow B = -1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\}$$

$$= 1 - t + \frac{\sin 2t}{2}$$

5) The gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du, \quad x > 0.$$

a) Show that for any real number $r > -1$, we have

$$\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}}, \quad s > 0.$$

$$\begin{aligned} \mathcal{L}\{t^r\} &= \int_0^\infty e^{-st} t^r dt \quad \text{let } u = st \\ &= \int_0^\infty e^{-u} \frac{u^r}{s^r} \frac{du}{s} \\ &= \frac{1}{s^{r+1}} \int_0^\infty e^{-u} u^r du = \frac{1}{s^{r+1}} \Gamma(r+1) \end{aligned}$$

b) Find $\mathcal{L}\{t^{-1/2}\}$ (Use $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.)

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$