

1) Solve the following initial value problem (IVP),

$$\frac{dy}{dx} = 1 + \cos^2(x - y), \quad y(1) = 1.$$

$$u = x - y \quad \frac{du}{dx} = 1 - \frac{dy}{dx}$$

$$1 - \frac{du}{dx} = 1 + \cos^2 u$$

$$\frac{1}{\cos^2 u} du = - dx$$

$$\sec^2 u du = - dx$$

$$\int \sec^2 u du = \int -dx$$

$$\tan u = -x + C$$

$$\tan(x - y) = -x + C$$

$$y(1) = 1 \Rightarrow \tan 0 = -1 + C \Rightarrow C = 1$$
$$-1 + C = 0$$

$$\tan(x - y) = -x + 1$$

or \$x - \arctan(-x) = y\$

2) Solve the equation,

$$\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x},$$

$$\frac{dy}{dx} = \frac{y}{x} \ln \frac{y}{x} + \frac{y}{x} \quad \text{homogeneous}$$

$$\text{Let } v = \frac{y}{x} \quad y = xv \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = v \ln v + v$$

$$\frac{dv}{v \ln v} = \frac{dx}{x}$$

$$\int \frac{dv}{v \ln v} = \int \frac{dx}{x}$$

$$\ln |\ln v| = \ln |x| + \ln c_1$$

$$|\ln v| = c_1 |x|$$

$$\ln v = \pm c_1 x \quad \text{replace } \pm c_1 \text{ by } c$$

$$\ln v = cx$$

$$v = e^{cx}$$

$$\frac{y}{x} = e^{cx}$$

$$y = x e^{cx}$$

3) Solve the equation,

$$(ye^{-2x} + y^3)dx - e^{-2x}dy = 0.$$

$$\frac{dy}{dx} = \frac{ye^{-2x} + y^3}{e^{-2x}} = y + e^{2x}y^3$$

$$\frac{dy}{dx} - y = e^{2x}y^3 \quad (\text{Bernoulli})$$

$$y^{-3} \frac{dy}{dx} - y^{-2} = e^{2x} \quad \text{let } \vartheta = y^{-2}$$

$$\frac{d\vartheta}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$-\frac{1}{2} \frac{d\vartheta}{dx} - \vartheta = e^{2x}$$

$$2 \frac{d\vartheta}{dx} + 2\vartheta = -2e^{2x} \quad \mu(x) = e^{\int 2dx} = e^{2x}$$

$$e^{2x} \frac{d\vartheta}{dx} + 2e^{2x}\vartheta = -2e^{4x}$$

$$\frac{d}{dx}(e^{2x}\vartheta) = -2e^{4x}$$

$$e^{2x}\vartheta = -\frac{1}{2}e^{4x} + C$$

$$\vartheta = -\frac{1}{2}e^{2x} + Ce^{-2x}$$

$$\boxed{y^{-2} = -\frac{1}{2}e^{2x} + Ce^{-2x}}$$

4) If $y_1(x) = \tan x$ is a solution of

$$y' = -y^2 + 3y \tan x - \tan^2 x + 1,$$

find the general solution.

$$y = \tan x + \frac{1}{\varphi} \quad y' = \sec^2 x - \frac{\varphi'}{\varphi^2}$$
$$\cancel{\sec^2 x} - \frac{\varphi'}{\varphi^2} = -\cancel{\tan^2 x} - \frac{1}{\varphi^2} - \frac{2\tan x}{\varphi} + \cancel{3\tan x} + \frac{3\tan x}{\varphi} - \cancel{\tan^2 x} + 1$$
$$-\frac{\varphi'}{\varphi^2} = -\frac{1}{\varphi^2} + \frac{\tan x}{\varphi}$$

$$\varphi' + (\tan x)\varphi = 1 \quad \mu(x) = e^{\int \tan x dx} = e^{-\ln \cos x} = \frac{1}{\cos x}$$

$$\frac{1}{\cos x} \varphi' + \frac{\tan x}{\cos x} \varphi = \frac{1}{\cos x}$$

$$\frac{d}{dx} \left(\frac{1}{\cos x} \varphi \right) = \frac{1}{\cos x}$$

$$\frac{1}{\cos x} \varphi = \ln |\sec x + \tan x| + C$$

$$\varphi = \cos x \ln |\sec x + \tan x| + C \cos x$$

$$y = \tan x + \frac{1}{\cos x \ln |\sec x + \tan x| + C \cos x}$$

5) Solve the equation.

$$(x+2y-4)dx - (2x+y-5)dy = 0.$$

$$\begin{aligned} x &= X+h \quad h+2k-4=0 \\ y &= Y+k \quad \underline{-2/2h+k-5=0} \\ &\quad -3h+6=0 \quad h=2 \quad k=1 \end{aligned}$$

$$\begin{cases} x = X+2 \\ y = Y+1 \end{cases} \Rightarrow (X+2Y)dx - (2X+Y)dy = 0$$

$$\frac{dy}{dx} = \frac{X+2Y}{2X+Y} = \frac{1+\frac{2Y}{X}}{2+\frac{Y}{X}} \quad \frac{y}{x} = v$$

$$v + X \frac{dv}{dx} = \frac{1+2v}{2+v^2}$$

$$X \frac{dv}{dx} = \frac{1+2v^2-2v-1-v^2}{2+v^2}$$

$$\frac{2+v^2}{1-v^2} dv = \frac{dx}{X}$$

$$\begin{aligned} \frac{2+v^2}{1-v^2} &= \frac{A}{v+1} + \frac{B}{v+1} = \frac{-(A+B)v^2 + B-A}{1-v^2} \\ A+B &= -1 \\ B-A &= 2 \quad B = 1/2 \\ A &= -3/2 \end{aligned}$$

$$-\frac{3}{2} \frac{1}{v-1} + \frac{1}{2} \frac{1}{v+1} dv = \frac{dx}{X}$$

$$\ln \left[\frac{(v+1)^{1/2}}{(v-1)^{3/2}} \right] = \ln x + \ln C_1$$

$$\sqrt{\frac{v+1}{(v-1)^3}} = C_1 X$$

$$\sqrt[3]{\frac{x+y}{(y-x)^3}} = C_1 X$$

$$C_1^2 = C$$

$$x+y = C (y-x)^3$$